Wavelets and Polarized Data on the Sphere

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TABLE OF CONTENT

- Wavelet/Curvelet on the Sphere
- Polarized Data
  - Euclidian Images
  - Spherical Images
- Restoration

PLANCK PROJECT

Successor of WMAP (better resolution, better sensitivity, more channels)

Launch in 2009

Two instruments LFI and HFI

Nine maps at 30,44,70,100,143,217,353,545,857 GHz

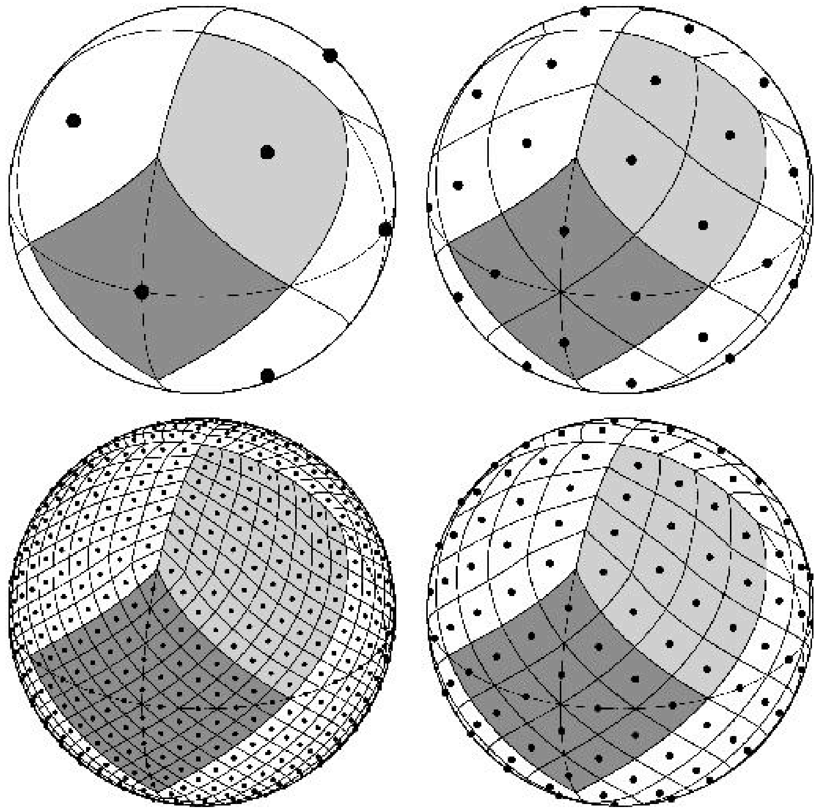
Angular resolutions: 33', 24', 14', 10', 7.1', 5', 5', 5', 5'

Size of each map = $9 \times 12 \times 2048^2$

Healpix

K.M. Gorski et al., 1999, astro-ph/9812350, http://www.eso.org/science/healpix

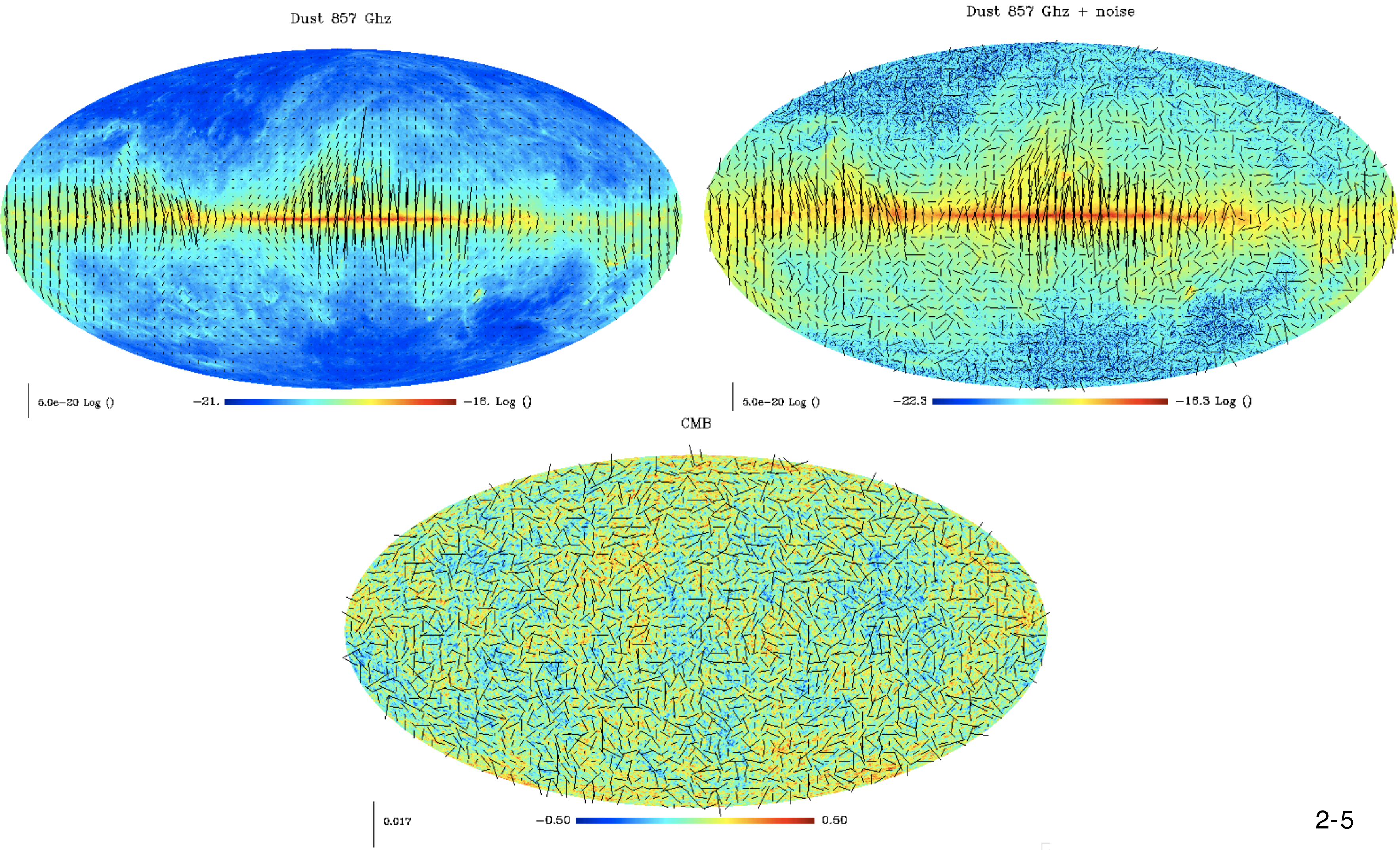
- Pixels = Rhombus
- Same Surfaces
- For a given latitude : regularly spaced
- Nbr of pixels :12*nside^2
- Includeds in the software:
  - Anafast
  - Synfast



Planck (9 channels): nside=2048 ==> Number of pixels = $2048^{\wedge 2}$ * 12 * 9

PLANCK POLARIZED DATA: T, Q, U

Magnitude $P = \sqrt{Q^2 + V^2}$ Orientation $\alpha = \arctan\left(U/Q\right)$



Why do we need sparse representations for Polarized Data ?

Non-Gaussianity statistical test (ex: Laurence Perotto talk)

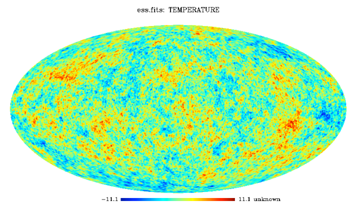
Detection (ex: Marcos Cruz talk)

Denoising/Deconvolution (ex: Yassir Moudden poster)

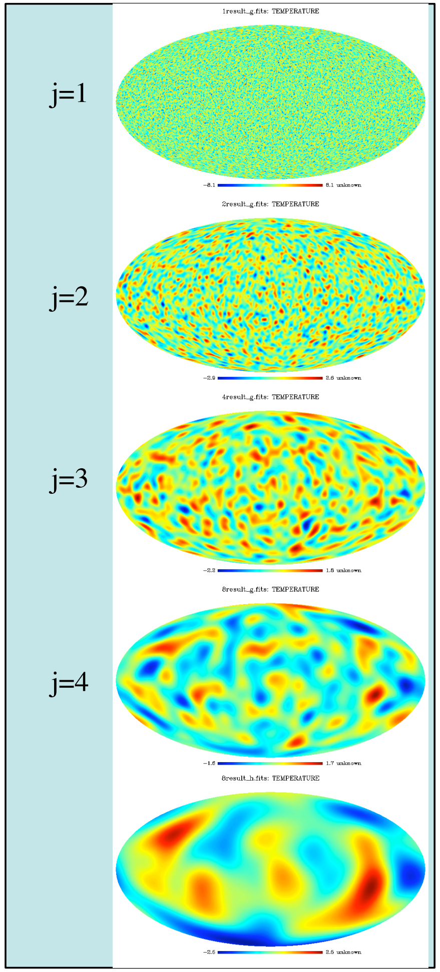
Component Separation (ex: Jerome Bobin talk)

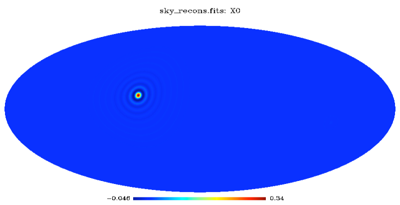
Mask problem ==> inpainting (ex: Jalal Fadili Talk & Sandrine Pires Poster)

Isotropic Undecimated Wavelet on the Sphere



Undecimated Wavelet Transform





$$\hat{\psi}_{\frac{l_c}{2^j}}(l,m) = \hat{\phi}_{\frac{l_c}{2^{j-1}}}(l,m) - \hat{\phi}_{\frac{l_c}{2^j}}(l,m)$$

$$\hat{H}_j(l,m) \quad = \quad \begin{cases} \dfrac{\hat{\phi}_{\frac{l_c}{2^{j+1}}}(l,m)}{\hat{\phi}_{\frac{l_c}{2^j}}(l,m)} & \text{if } l < \frac{l_c}{2^{j+1}} \quad \text{and} \quad m = 0 \\ 0 & \text{otherwise} \end{cases}$$

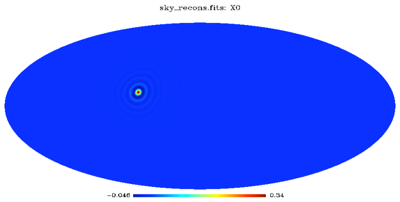
$$\hat{G}_j(l,m) \quad = \quad \begin{cases} \dfrac{\hat{\psi}_{\frac{l_c}{2^{j+1}}}(l,m)}{\hat{\phi}_{\frac{l_c}{2^j}}(l,m)} & \text{if } l < \frac{l_c}{2^{j+1}} \quad \text{and} \quad m = 0 \\ 1 & \text{if } l \geq \frac{l_c}{2^{j+1}} \quad \text{and} \quad m = 0 \\ 0 & \text{otherwise} \end{cases}$$

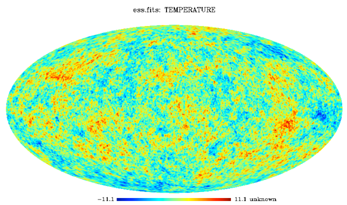
$$\hat{c}_{j+1}(l,m) = \widehat{H}_j(l,m)\hat{c}_j(l,m)$$

$$\hat{w}_{j+1}(l,m) = \widehat{G}_j(l,m)\hat{c}_j(l,m)$$

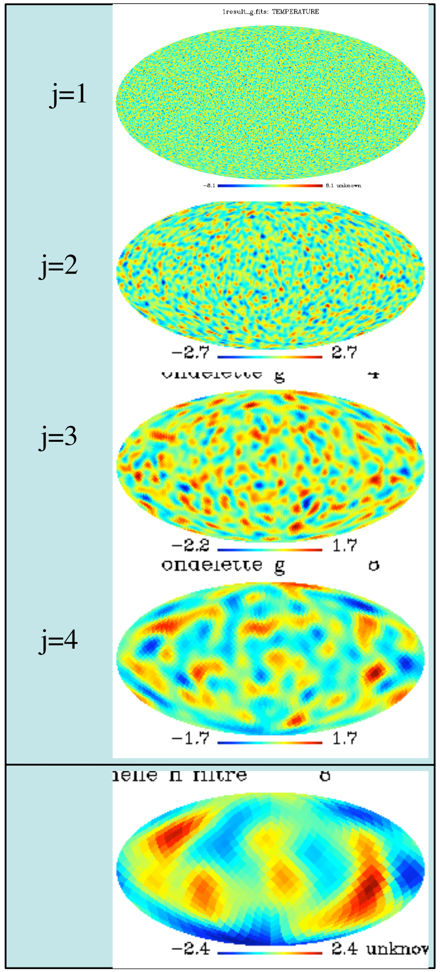
$$\boxed{c_0(\vartheta,\varphi) = c_J(\vartheta,\varphi) + \sum_{j=1}^{J} w_j(\vartheta,\varphi)}$$

Isotropic Pyramidal Wavelet on the Sphere





Pyramidal
Wavelet Transform



$$\hat{\psi}_{\frac{l_c}{2^j}}(l,m) = \hat{\phi}_{\frac{l_c}{2^{j-1}}}(l,m) - \hat{\phi}_{\frac{l_c}{2^j}}(l,m)$$

$$\hat{c}_{j+1}(l,m) = \widehat{H}_j(l,m)\hat{c}_j(l,m)$$

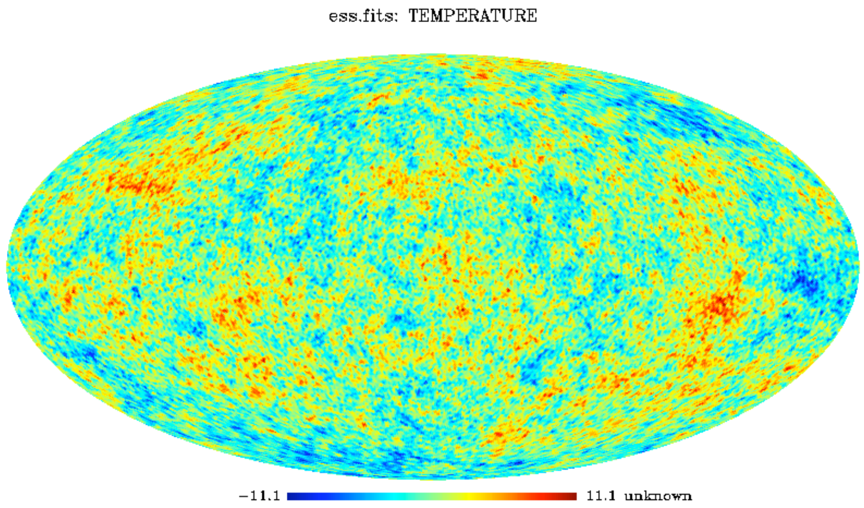
$$\hat{w}_{j+1}(l,m) = \widehat{G}_j(l,m)\hat{c}_j(l,m)$$

$$\widehat{\tilde{H}}_j = \sqrt{\frac{4\pi}{2l+1}}\hat{\tilde{h}}_j \quad = \widehat{H}_j^* / (\mid \widehat{H}_j \mid^2 + \mid \widehat{G}_j \mid^2)$$

$$\widehat{\tilde{G}}_j = \sqrt{\frac{4\pi}{2l+1}}\hat{\tilde{g}}_j \quad = \widehat{G}_j^* / (\mid \widehat{H}_j \mid^2 + \mid \widehat{G}_j \mid^2)$$

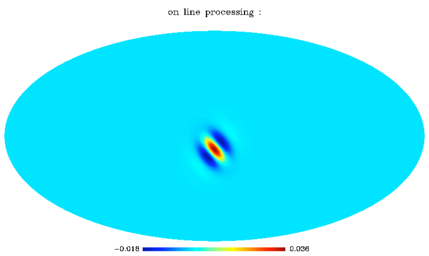
$$\boxed{\hat{c}_j = \hat{c}_{j+1}\widehat{\tilde{H}}_j + \hat{w}_{j+1}\widehat{\tilde{G}}_j}$$

Wavelet, Ridgelet and Curvelet on the Sphere :



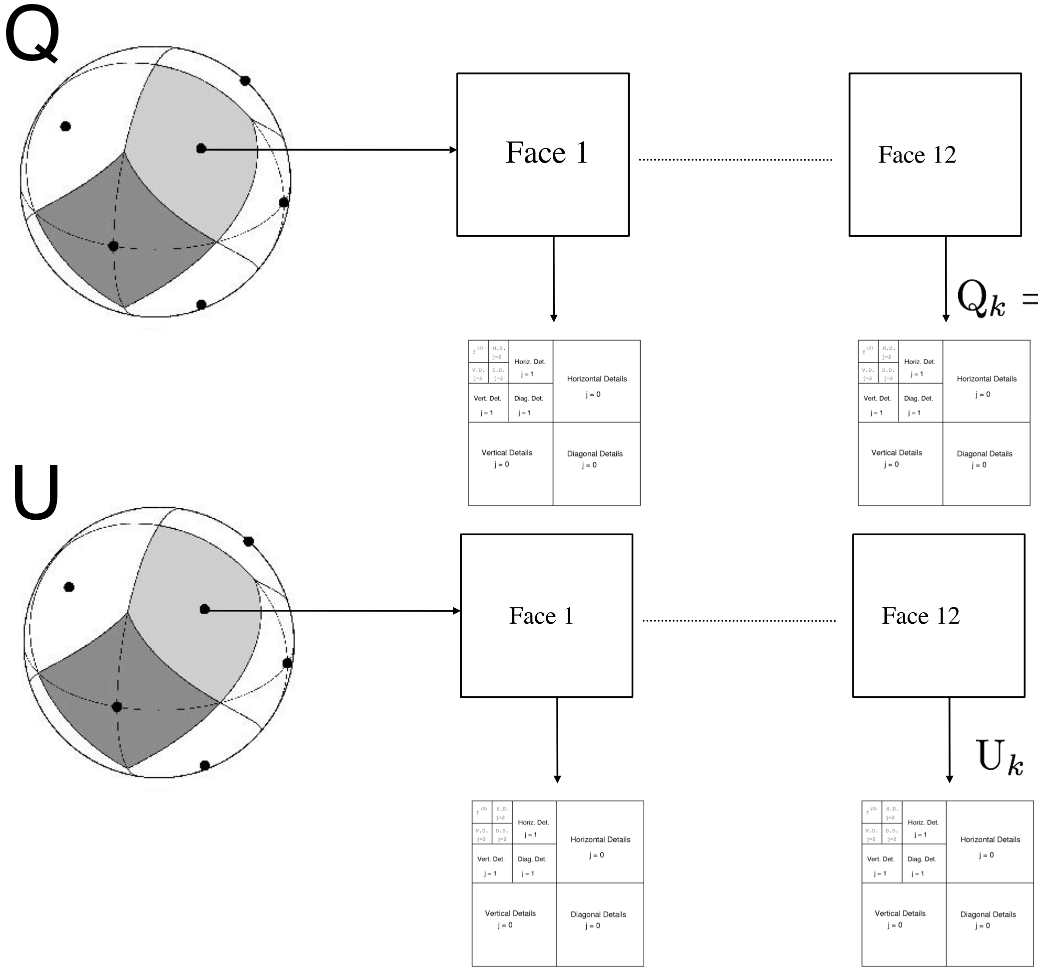
Wavelets, Ridgelets and Curvelets on the Sphere, *Astronomy & Astrophysics*, 446, 1191-1204, 2006.

Software available at: http://jstarck.free.fr/mrs.html



Multiscale transforms, Gaussianity tests
Denoising using Wavelets and Curvelets
Astrophysical Component Separation (ICA on the Sphere)

Orthogonal Q-U Polarized Wavelet on the Sphere

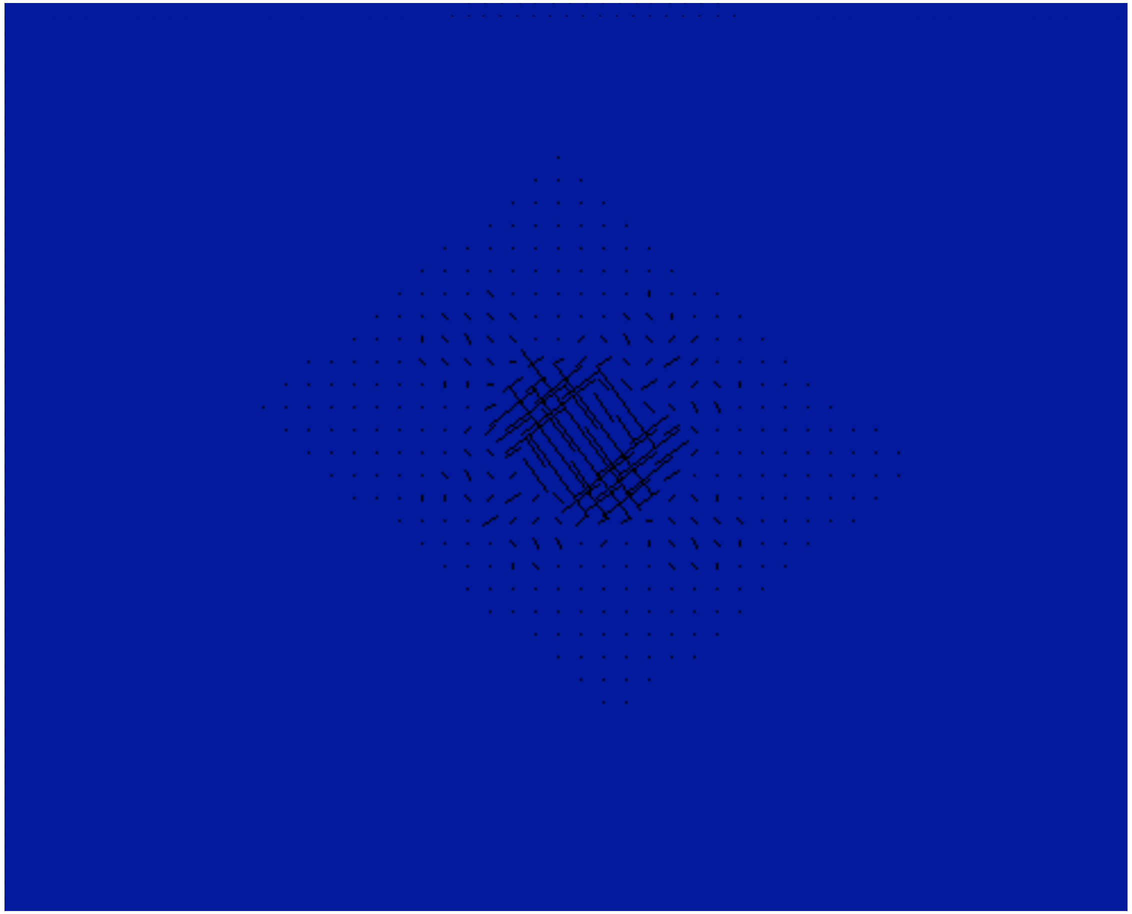


$$\mathrm{Q}_k = \sum_p c^Q_{J,p} \phi_{j,k}(p) + \sum_p \sum_{j=1}^{J} \psi_{j,k}(p) w^Q_{j,p}$$

$$\mathrm{U}_k = \sum_p c^U_{J,p} \phi_{j,k}(p) + \sum_p \sum_{j=1}^{J} \psi_{j,k}(p) w^U_{j,p}$$

$$(\mathrm{Q} \pm iU)_k = \sum_p (c^Q_{J,p} \pm c^U_{J,p}) \phi_{j,k}(p) + \sum_p \sum_{j=1}^{J} \psi_{j,k}(p) (w^Q_{j,p} \pm w^U_{j,p})$$

Q,U Orthogonal Wavelet Decomposition



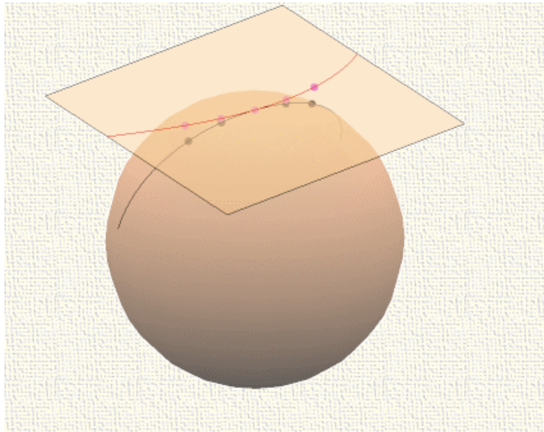
MP-Nonlinear Multiscale Transform
Multiscale Representation of Module-Phase instead of Q-U

$$\mathrm{Q} + \mathrm{iU} = \boxed{\mathrm{P}}\ \boxed{\exp i\theta}$$

Wavelet/Curvelet Transform

Decimated NonLinear Multiscale Transform on S^1 manifold

Donoho, Drori, Schroeder, and Ur Rahman, SIAM Journal on Multiscale Modeling and Simulation, Vol 4, No. 4, 2005.

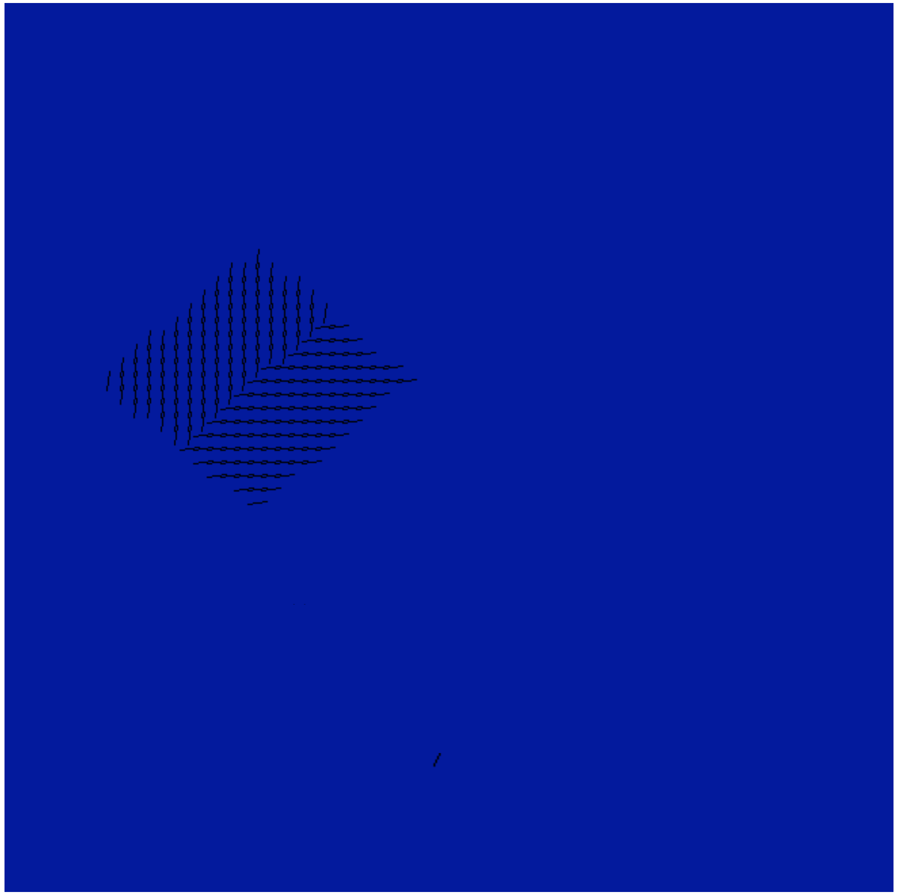


Interpolation-Refinement Scheme

$$\mathrm{c}_{j+1,k} = c_{j,2k}$$
$$\mathrm{w}_{j+1,k} = c_{j,2k+1} - \mathrm{Interp}_{2k+1}(c_{j+1})$$

Restoration Problem ⇒ Undecimated MP-Multiscale Transform

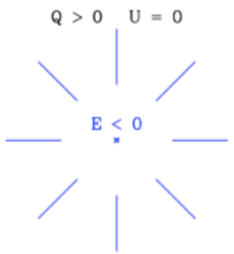
Decimated Wavelet Transform on a Manifold



E/B Mode Decomposition

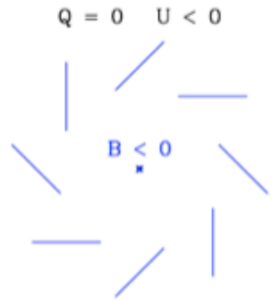
$$E = \sum_{\ell,m} a^E_{\ell m} Y_{\ell m} = \sum_{\ell,m} -\frac{{}_2a_{\ell m} + {}_{-2}a_{\ell m}}{2} Y_{\ell m} \qquad a^E_{lm} = -(a_{2,lm} + a_{-2,lm})/2$$

$$B = \sum_{\ell,m} a^B_{\ell m} Y_{\ell m} = \sum_{\ell,m} i\frac{{}_2a_{\ell m} - {}_{-2}a_{\ell m}}{2} Y_{\ell m} \qquad a^B_{lm} = i(a_{2,lm} - a_{-2,lm})/2$$



$$Q = -\sum_{l,m}(a^E_{l,m} Z^+_{l,m} + i a^B_{l,m} Z^-_{l,m})$$

$$U = -\sum_{l,m}(a^B_{l,m} Z^+_{l,m} - i a^E_{l,m} Z^-_{l,m})$$



$$Z^+_{l,m} = ({}_2Y_{l,m} + {}_{-2}Y_{l,m})/2$$

$$Z^-_{l,m} = ({}_2Y_{l,m} - {}_{-2}Y_{l,m})/2$$

E and B mode are closely related to the curl-free and div-free components of the vector field

E/B Undecimated Wavelet Transform for Polarized Data

Proceedings of SPIE , Vol 6701, D. Van De Ville, V. Goyal and M. Papadakis Editors, 2007.

$$
\begin{aligned}
E = & \sum_{\ell,m} a_{\ell m}^{E} Y_{\ell m} = \sum_{\ell,m} - \frac{{}_2a_{\ell m} + {}_{-2}a_{\ell m}}{2} Y_{\ell m} \\
B = & \sum_{\ell,m} a_{\ell m}^{B} Y_{\ell m} = \sum_{\ell,m} i \frac{{}_2a_{\ell m} - {}_{-2}a_{\ell m}}{2} Y_{\ell m}
\end{aligned}
$$

Wavelet Transform of E and B are obtained by:

$$w_j^E = < E, \psi_j > \qquad\qquad w_j^B = < B, \psi_j >$$

Furthermore, if we use the spherical isotropic wavelet construction of (starck et al, 2006), we have

$$E(\theta, \phi) = c_J^E(\theta, \phi) + \sum_{j=1}^{J} w_j^E(\theta, \phi) \qquad\qquad B(\theta, \phi) = c_J^B(\theta, \phi) + \sum_{j=1}^{J} w_j^B(\theta, \phi)$$

E/B Undecimated Wavelet Reconstuction

$$Q + iU = \sum_{lm} a_{2,lm}\ {}_2Y_{lm} \qquad Q - iU = \sum_{lm} a_{-2,lm}\ {}_{-2}Y_{lm}$$

$$Q = -\frac{1}{2}\sum_{\ell,m} a^E_{\ell m}({}_2Y_{\ell m} + {}_{-2}Y_{\ell m}) + i a^B_{\ell m}({}_2Y_{\ell m} - {}_{-2}Y_{\ell m}) = \sum_{\ell,m} a^E_{\ell m} Z^+_{\ell m} + i a^B_{\ell m} Z^-_{\ell m}$$

$$U = -\frac{1}{2}\sum a^B_{\ell m}({}_2Y_{\ell m} + {}_{-2}Y_{\ell m}) - i a^E_{\ell m}({}_2Y_{\ell m} - {}_{-2}Y_{\ell m}) = \sum_{\ell,m} a^B_{\ell m} Z^+_{\ell m} - i a^E_{\ell m} Z^-_{\ell m}$$

As we have: $E = c^E_J + \sum_{j=1}^{J} w^E_j$ and $B = c^B_J + \sum_{j=1}^{J} w^B_j$

Then
$$Q(\theta,\phi) = \sum_{l,m} c^E_{J,l,m} Z^+_{l,m} + i c^B_{J,l,m} Z^-_{l,m} + \sum_j \sum_{l,m} w^E_{j,l,m} Z^+_{l,m} + i w^B_{j,l,m} Z^-_{l,m}$$

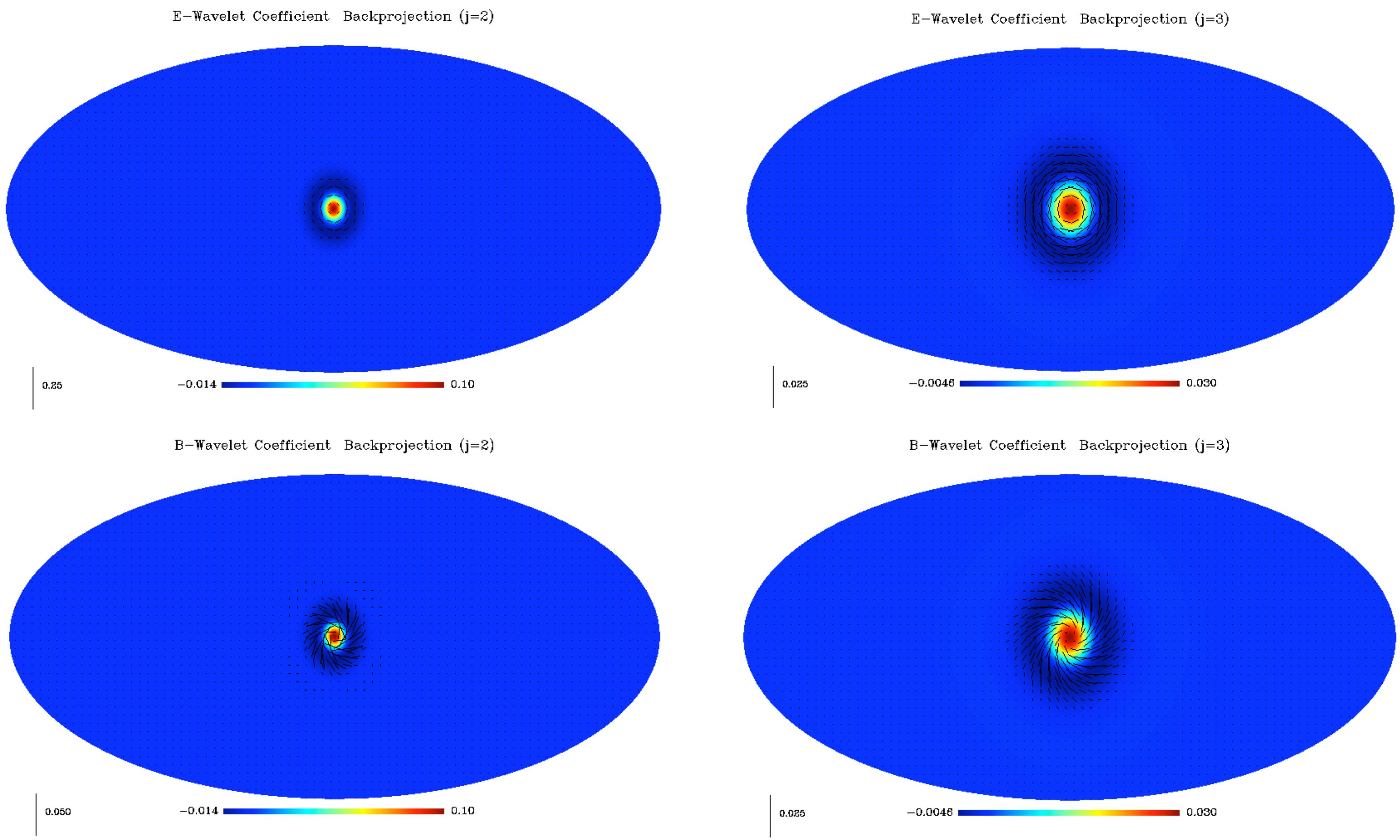
$$U(\theta,\phi) = \sum_{l,m} c^B_{J,l,m} Z^+_{l,m} - i c^E_{J,l,m} Z^-_{l,m} + \sum_j \sum_{l,m} w^B_{j,l,m} Z^+_{l,m} - i w^E_{j,l,m} Z^-_{l,m}$$

$$Q = \quad c_J^{E,+} + i c_J^{B,-} + \sum_{j=1}^{J} \left\{ w_j^{E,+} + i w_j^{B,-} \right\}$$

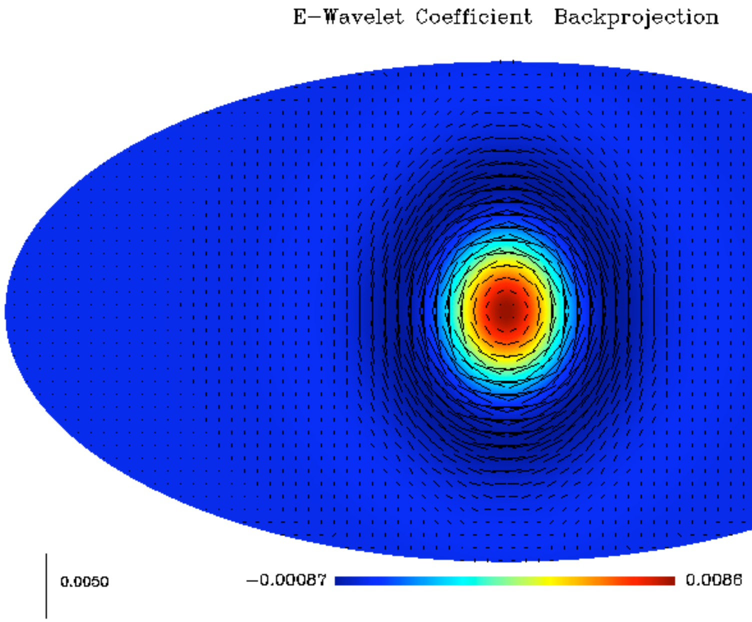
$$U = \quad c_J^{B,+} - i c_J^{E,-} + \sum_{j=1}^{J} \left\{ w_j^{B,+} - i w_j^{E,-} \right\}$$

$$c_J^{X,+} = c_J^X \sum_{\ell,m} Y^\dagger_{\ell m} Z^+_{\ell m} \quad \text{and} \quad c_J^{X,-} = c_J^X \sum_{\ell,m} Y^\dagger_{\ell m} Z^-_{\ell m}$$

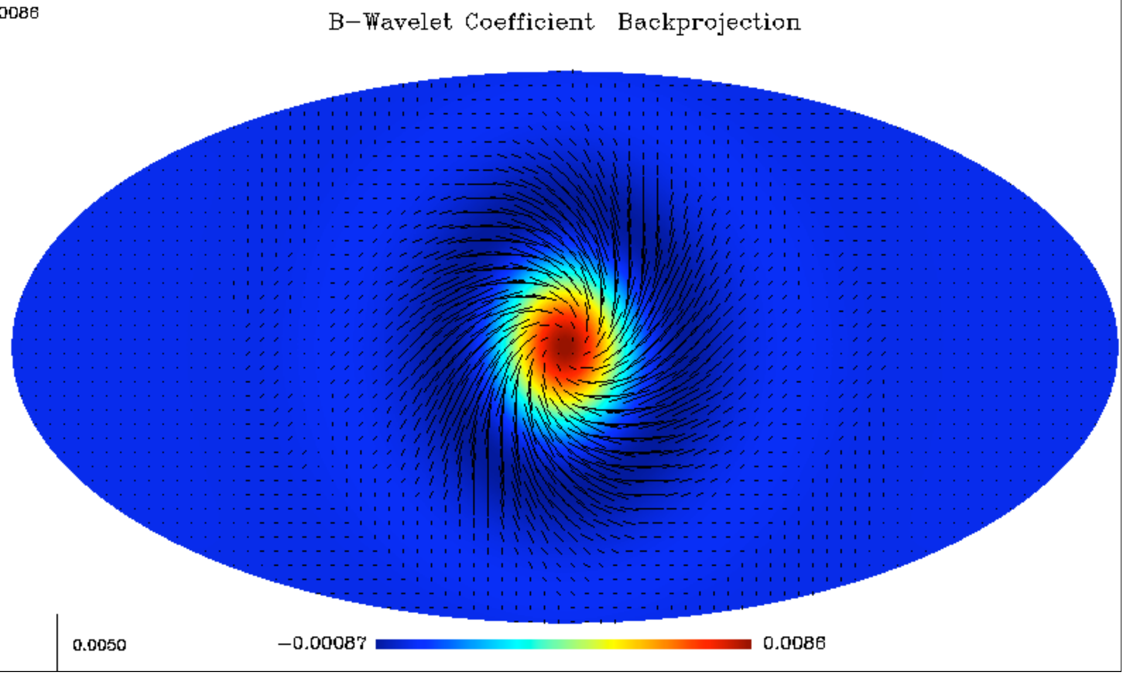
Wavelet and E/B Mode Decomposition



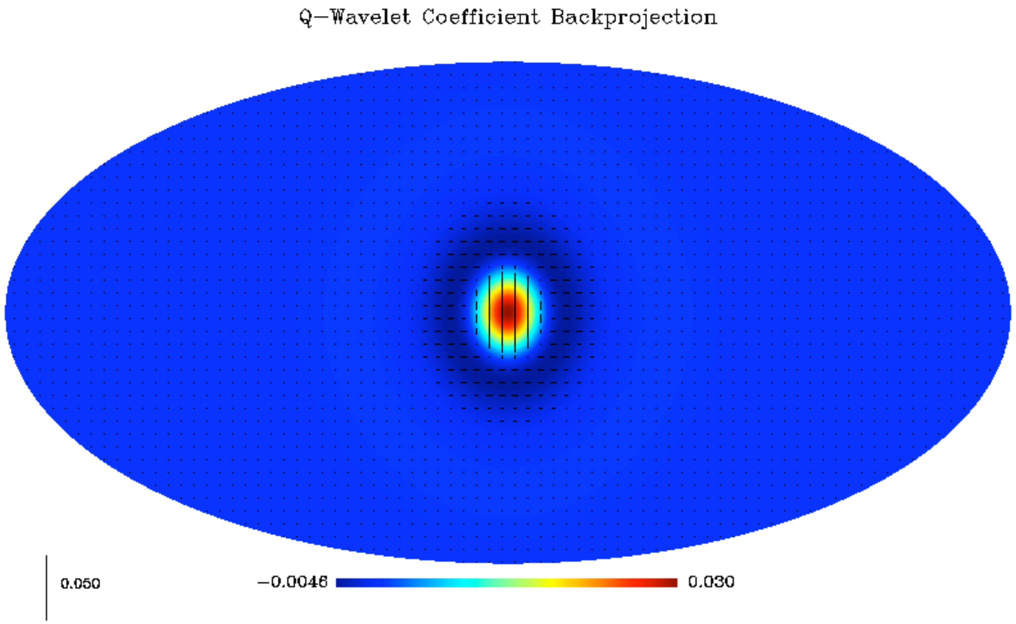
Wavelet and E/B Mode Decomposition

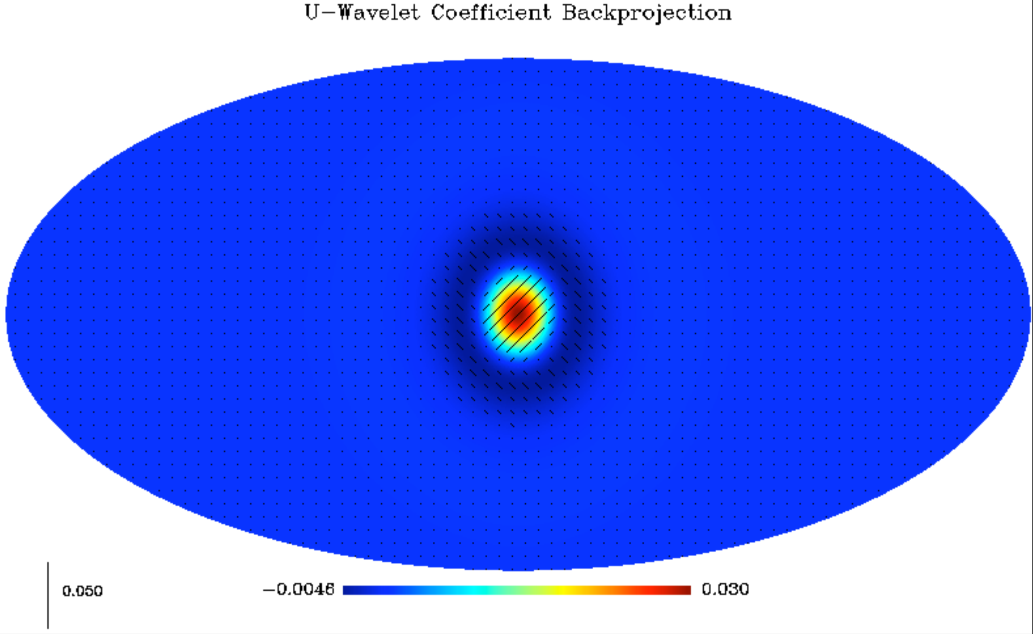


j=4

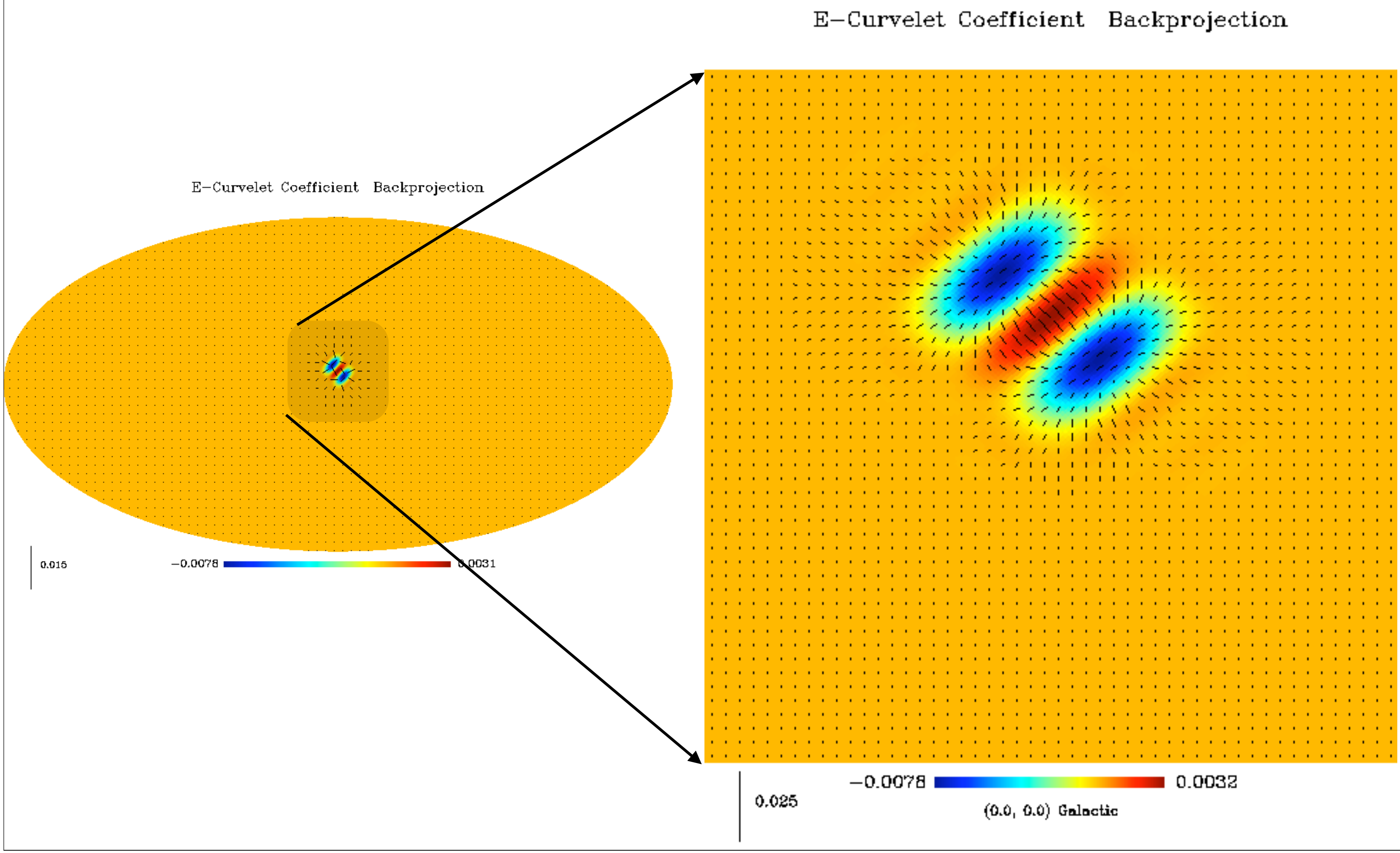


Q,U Isotropic Undecimated Wavelet Transform

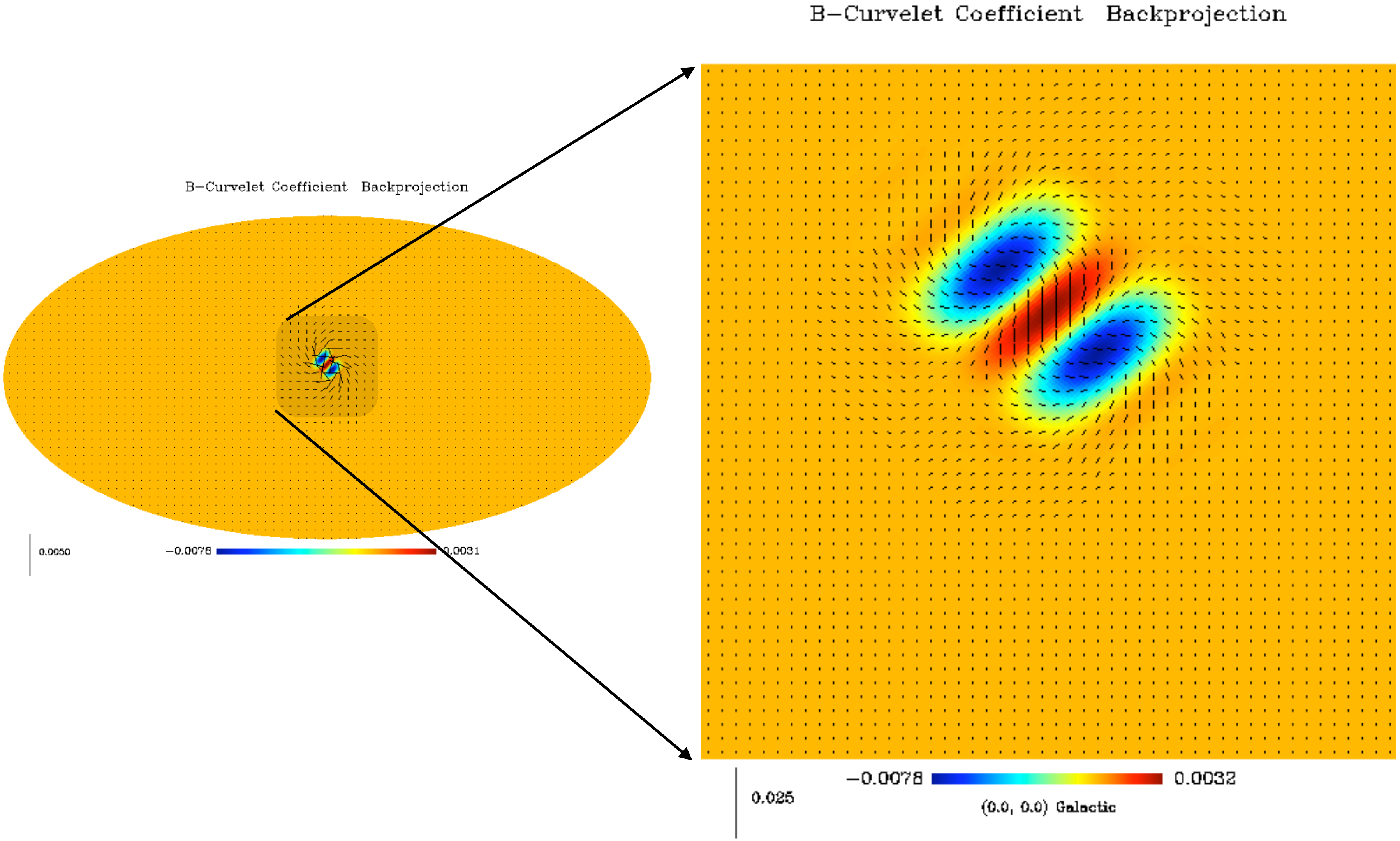




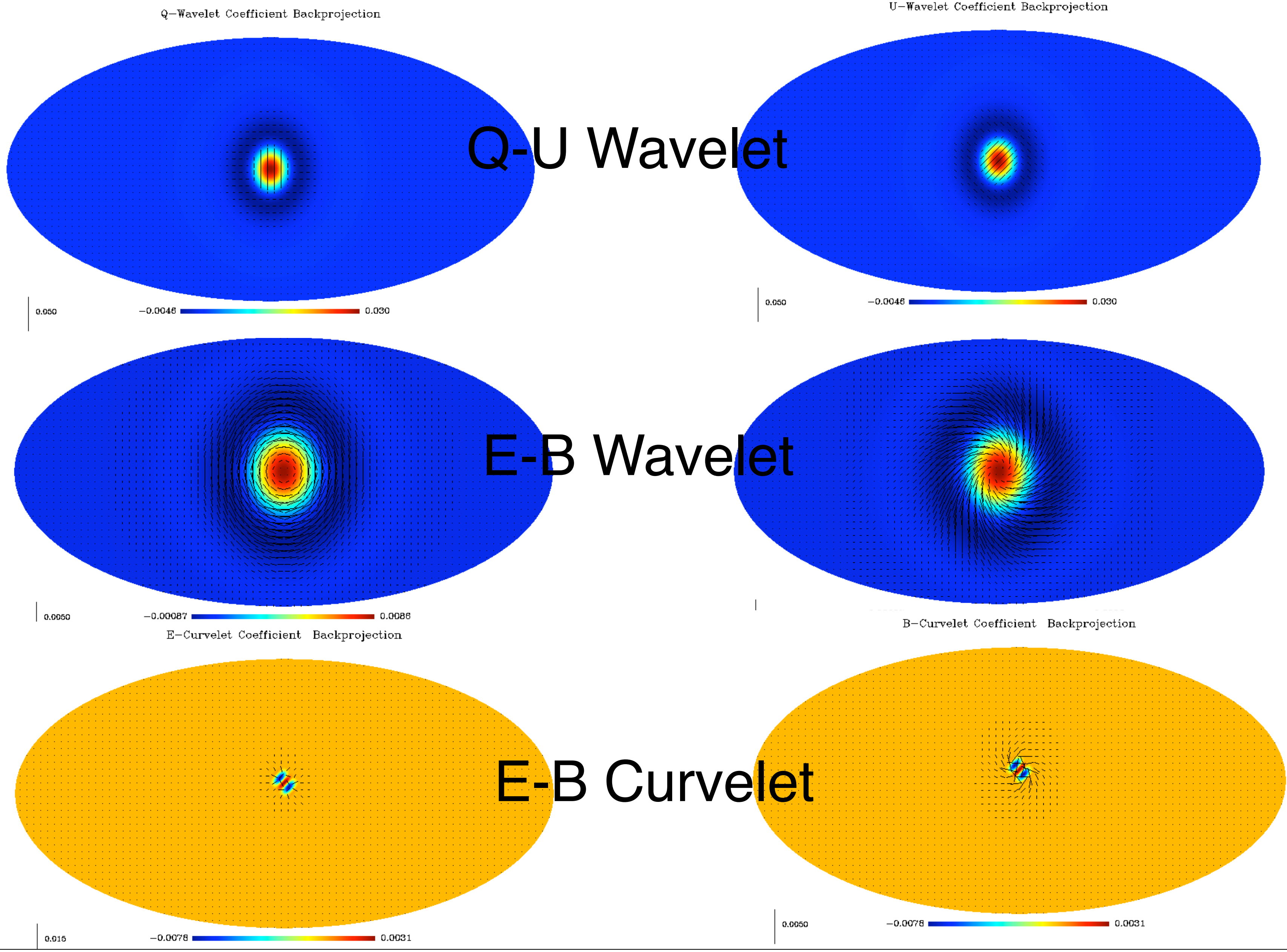
Curvelet and E/B Mode Decomposition



Curvelet and E/B Mode Decomposition



Sparse Represention on the Sphere and Polarization



Polarized Data Denoising

$$Q(\theta,\phi) = \sum_{l,m} c^E_{J,l,m} Z^+_{l,m} + i c^B_{J,l,m} Z^-_{l,m} + \sum_j \sum_{l,m} \tilde{w}^E_{j,l,m} Z^+_{l,m} + i \tilde{w}^B_{j,l,m} Z^-_{l,m}$$

$$U(\theta,\phi) = \sum_{l,m} c^B_{J,l,m} Z^+_{l,m} - i c^E_{J,l,m} Z^-_{l,m} + \sum_j \sum_{l,m} \tilde{w}^B_{j,l,m} Z^+_{l,m} - i \tilde{w}^E_{j,l,m} Z^-_{l,m}$$

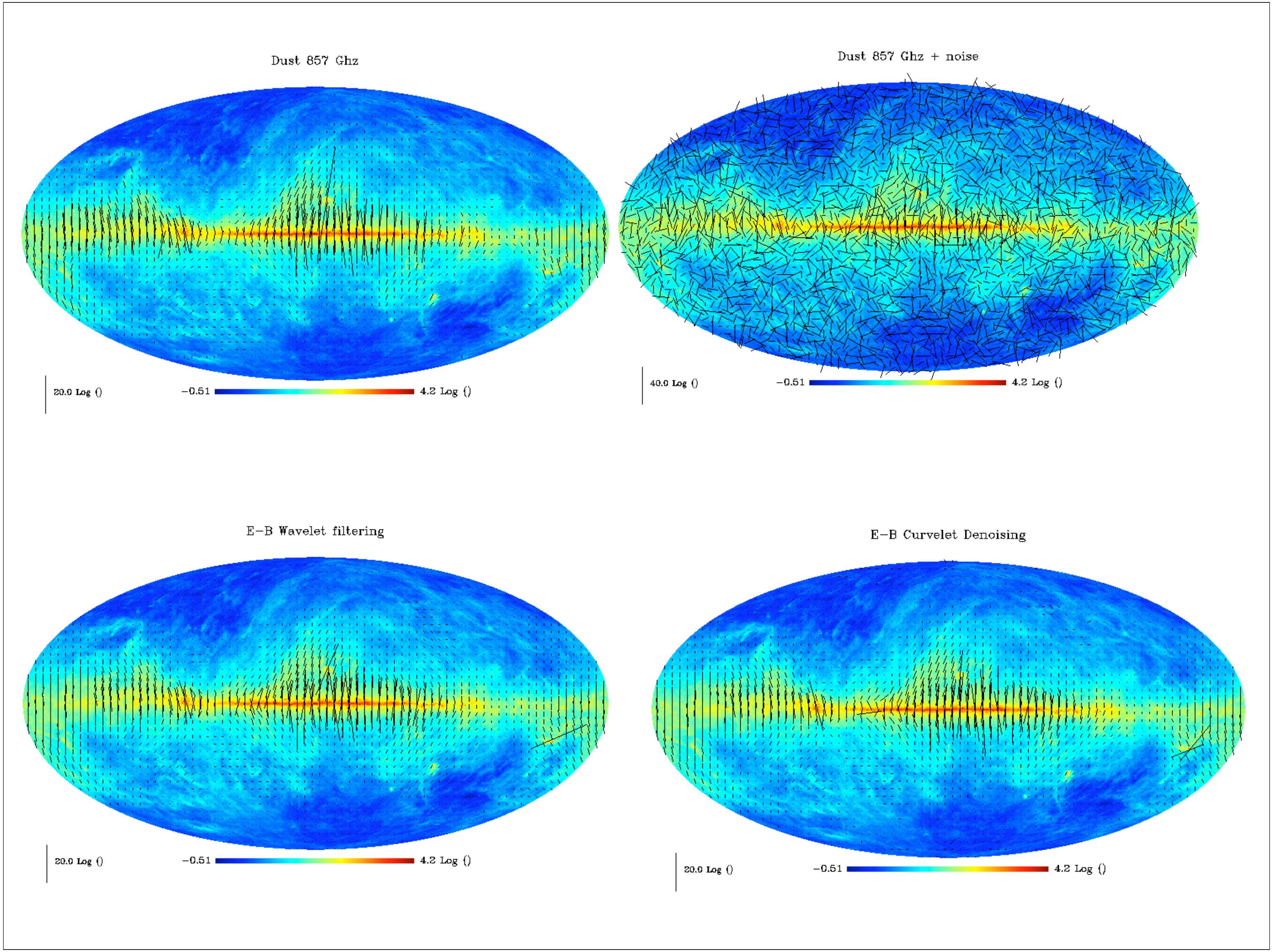
Where

$$\tilde{w}^E_{j,k} = \delta(w^E_{j,k})$$

$$\tilde{w}^B_{j,k} = \delta(w^B_{j,k})$$

Hard thresholding corresponds to the following non linear operation:

$$\tilde{w}_{j,k} = \begin{cases} w_{j,k} & \text{if } \mid w_{j,k} \mid \geq T_j \\ 0 & \text{otherwise} \end{cases}$$


Dust 857 Ghz
20.0 Log ()
−0.51
4.2 Log ()
Dust 857 Ghz + noise
40.0 Log ()
−0.51
4.2 Log ()
E−B Wavelet filtering
20.0 Log ()
−0.51
4.2 Log ()
E−B Curvelet Denoising
20.0 Log ()
−0.51
4.2 Log ()

Conclusions

- We have developed new sparse decompositions for PLANCK polarized data:
  On patches:
  - Decimated Mod-Phase Wavelet transform
  - Undecimated Mod-Phase Wavelet transform

  On the sphere:
  - (QU-EB) - Undecimated Wavelet
  - (QU-EB) - Pyramidal Wavelet
  - (QU-EB) - Curvelet
- Preliminary result for denoising applications

- **Perspectives**:

NOISE

- ★ Module-Phase transform ==> multiplicative noise
- ★ Correlated noise
- ★ Thresholding strategy: independent or joint threshold ?
- ★ Hard thresholding ==> more sophisticated threshold ? (ex: Panos talk)

Power-spectrum & Missing data (ex: Jalal Fadili talk)